ON THE RESTRICTED CESÀRO SUMMABILITY OF DOUBLE FOURIER SERIES

BY

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1. We suppose throughout this paper that $\phi(u, v) \in L(0, 0; \pi, \pi)$ and is periodic, with period 2π, and that

$$(1.1) \qquad \phi(u, v) \sim \sum_{m,n=0}^{\infty} a_{mn} \cos mu \cos nv.$$

We denote $\sum_{m,n=0}^{\infty} a_{mn}$ by $S[\phi]$ and write

$$(1.2) \qquad \sigma_{mn}^{\alpha,\beta} = (A_m^\alpha A_n^\beta)^{-1} \sum_{r=0}^{m} \sum_{s=0}^{n} A_{m-r}^\alpha A_{n-s}^\beta a_{rs} = (A_m^\alpha A_n^\beta)^{-1} S_{mn}^{\alpha,\beta},$$

where

$$A_m^\alpha = \binom{m+\alpha}{m}.$$

We also write $u^a v^b \phi_{a,b}(u, v)$ for the fractional integral of order (a, b) $(a \geqq 0, b \geqq 0)$ of $\phi(u, v)$, so that, in particular, $\phi_{0,0}(u, v) = \phi(u, v)$,

$$(1.3) \quad \phi_{a,b}(u, v) = ab u^{-a} v^{-b} \int_0^u \int_0^v (u - x)^{a-1} (v - y)^{b-1} \phi(x, y) dx dy \qquad (a > 0,\ b > 0),$$

and $\phi_{0,b}(u, v)$, $\phi_{a,0}(u, v)$ are interpreted in the natural way (cf. [4, p. 413]).

The problem of the convergence, in some sense, of the means $\sigma_{mn}^{\alpha,\beta}$, and its connexion with the behaviour of the functional means $\phi_{a,b}(u, v)$, has been considered by a number of writers. Gergen and Littauer [4, Theorems IV and V] have treated the problem of the boundedness, and convergence in the Pringsheim sense, of $\sigma_{mn}^{\alpha,\beta}$. They also considered the corresponding problem when the restriction of boundedness on $\sigma_{mn}^{\alpha,\beta}$ is removed and proved the following theorem.

THEOREM A. *If* $a-2>\alpha\geqq 0$, $b-2>\beta\geqq 0$, *if* $\phi_{a,b}(u, v)$ *is bounded in* $(0, 0; \delta, \delta)$ *for some positive* δ, *and if* $\sigma_{mn}^{\alpha,\beta} \to s$ *as* $(m, n) \to (\infty, \infty)$, *then* $\phi_{a,b}(u, v) \to s$ *as* $(u, v) \to (+0, +0)$.

The question of whether a "converse" of this theorem is true; i.e., whether for suitably related a, b, α, β, $\phi_{a,b}(u, v) \to s$ together with boundedness of $\sigma_{mn}^{\alpha,\beta}$

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for large m and n imply that $\sigma_{mn}^{\alpha,\beta}\to s$ as $(m, n)\to(\infty, \infty)$, was left unanswered. Later Gergen [3, Theorem IV] showed that it is not possible to obtain such a theorem and proved instead [3, Theorem V] the following result which contains a "mixed" boundedness condition.

THEOREM B. *If* $0\leqq a<\xi$, $0\leqq\alpha<\xi-1$; $0\leqq b<\eta$, $0\leqq\beta<\eta-1$; *if* $\sigma_{mn}^{\alpha,\eta}$, $\sigma_{mn}^{\xi,\beta}$ *are bounded for large* m *and* n *and if* $\phi_{a,b}(u, v)\to s$ *as* $(u, v)\to(+0, +0)$, *then* $\sigma_{mn}^{\alpha,\beta}\to s$ *as* $(m, n)\to(\infty, \infty)$.

These results may be regarded as extensions to double series of well-known theorems of Paley [8] and Bosanquet [1].

A problem of a different character arises if we consider the convergence of $\sigma_{mn}^{\alpha,\beta}$ *in a restricted sense* instead of in the Pringsheim sense. A double sequence $\{b_{mn}\}$ is said to converge restrictedly to s, in symbols $b_{mn}\to s(R)$ as $(m, n)\to(\infty, \infty)$, if, for every $\lambda\geqq 1$, $b_{mn}\to s$ as $(m, n)\to(\infty, \infty)$ in such a way that $\lambda^{-1}\leqq mn^{-1}\leqq\lambda$. If $\sigma_{mn}^{\alpha,\beta}\to s(R)$ as $(m, n)\to(\infty, \infty)$ we shall say that $S[\phi]$ is summable $(C; \alpha, \beta)(R)$ to s. The concept of restricted summability was introduced by Moore [7] who proved the following theorem.

THEOREM C. *If* $\phi(u, v)\to s$ *as* $(u, v)\to(+0, +0)$ *then* $S[\phi]$ *is summable* $(C; \alpha, \beta)(R)$ *to* s *whenever* $\alpha\geqq 1$, $\beta\geqq 1$.

The present paper consists of an elaboration of the observation that the conclusion of Theorem C holds if we replace the hypothesis by restricted continuity and local boundedness, of $\phi(u, v)$ at $(0, 0)$. More precisely: we shall say that $\phi(u, v)\to s(C; a, b)(R)$ as $(u, v)\to(+0, +0)$ if, for any $\lambda\geqq 1$, $\phi_{a,b}(u, v)\to s$ as $(u, v)\to(+0, +0)$ in such a way that $\lambda^{-1}\leqq uv^{-1}\leqq\lambda$, and we prove the following two theorems.

THEOREM 1. *If* $\alpha\geqq 1$, $\beta\geqq 1$; $\alpha>a\geqq 0$, $\beta>b\geqq 0$; *if* $\phi(u, v)\to s(C; a, b)(R)$, *and if* $\phi_{a,b}(u, v)$ *is bounded in* $(0, 0; \delta, \delta)$ *for some positive* δ, *then* $S[\phi]$ *is summable* $(C; \alpha, \beta)(R)$ *to* s.

THEOREM 2. *If* $a-2>\alpha\geqq 0$, $b-2>\beta\geqq 0$; *if* $S[\phi]$ *is summable* $(C; \alpha, \beta)(R)$ *to* s, *and, if, for some* N, $\sigma_{mn}^{\alpha,\beta}$ *is bounded for* $m>N$, $n>N$, *then* $\phi(u, v)\to s(C; a, b)(R)$ *as* $(u, v)\to(+0, +0)$.

Theorem 1 (which contains Theorem C), and Theorem 2 may also be regarded as extensions of the results of Paley and Bosanquet. Before going on to prove these theorems we mention two noteworthy facts which suggest that, although summability $(C; \alpha, \beta)(R)$ is not a regular method, its application to double Fourier series has some advantages over the method used in Theorems A and B. Firstly, Herriot ([5], cf. Lemma 1 below) has shown that the summability $(C; \alpha, \beta)(R)$ $(\alpha\geqq 1, \beta\geqq 1)$ of $S[\phi]$ depends only on the behaviour of $\phi(u, v)$ near $(0, 0)$. Secondly, Zygmund [10, p. 309] has shown that the series in (1.1) is summable $(C; \alpha, \beta)(R)$ $(\alpha\geqq 1, \beta\geqq 1)$ to $\phi(u, v)$ for almost every $(u, v) \in (0, 0; \pi, \pi)$. Both these results reflect well-known (C)-summability properties of single Fourier series and neither holds for summability $(C; \alpha, \beta)$ interpreted as involving the existence of $\lim_{(m,n)\to(\infty,\infty)} \sigma_{mn}^{\alpha,\beta}$ in the Pringsheim sense, whether or not a boundedness condition is imposed (cf. [10, p. 304; 9]).

2. We first give some further notation, collect some known results and establish three lemmas.

If $\{b_{mn}\}$ is a given double sequence, if $\lambda \geqq 1$, and if

$$\overset{(\lambda)}{\alpha_{mn}} = \begin{cases} d_{mn} & \text{for } \lambda^{-1} \leqq mn^{-1} \leqq \lambda \\ 0 & \text{otherwise} \end{cases}$$

then we define

$$\lambda - \sup_{(m,n)} b_{mn} = \sup_{m\geqq 0, n\geqq 0} \overset{(\lambda)}{d_{mn}},$$

$$\lambda - \lim_{(m,n)} b_{mn} = \lim_{(m,n)\to(\infty,\infty)} \overset{(\lambda)}{d_{mn}},$$

$$\lambda - \limsup_{(m,n)} b_{mn} = \limsup_{(m,n)\to(\infty,\infty)} \overset{(\lambda)}{d_{mn}}.$$

In a similar way, if $f(u, v)$ is a given function, defined for $u>0$, $v>0$, then for $\lambda \geqq 1$ we define

$$\lambda - \lim_{(u,v)} f(u, v) = \lim_{(u,v)\to(+0,+0)} g_\lambda(u, v)$$

where $g_\lambda(u, v) = f(u, v)$ if $\lambda^{-1} \leqq uv^{-1} \leqq \lambda$ and is zero otherwise. We define $\lambda - \sup_{(u,v)} f(u, v)$ and $\lambda - \lim \sup_{(u,v)} f(u, v)$ in a similar way.

We shall require the functions

$$K_\alpha(m, u) = \frac{1}{2} + (A_m^\alpha)^{-1} \sum_{r=1}^{m} A_{m-r}^\alpha \cos ru$$

which are known [2, p. 64] to satisfy

$$(2.1) \qquad \left| K_\alpha^{(r)}(m, u) \right| \leqq \begin{cases} Am^{r+1} \\ Am^{r+1} \ \max[(mu)^{-r-2}, (mu)^{-\alpha-1}], \end{cases}$$

for $\alpha>0$, $r=0, 1, \cdots$, and $0 \leqq u \leqq \pi$; where A is independent of m and u. It follows easily from (2.1) that for $r=0, 1, \cdots$ and $m \geqq 0$

$$(2.2) \qquad \int_0^\pi u^r \left| K_\alpha^{(r)}(m, u) \right| du < A \qquad (\alpha > r)$$

where A is independent of m.

We also require Young's functions $\gamma_p(t)$ defined by

$$\gamma_p(t) = p\int_0^1 (1-u)^{p-1}\cos ut\, du \qquad (p>0).$$

It is known [2, p. 64] that, for $p>0$, $k=0, 1, \cdots; n=0, 1, \cdots$ and $t\geqq 0$

$$(2.3)\qquad |\Delta^k\gamma_p(nt)| \leqq \begin{cases} At^k \\ At^k \max[(nt)^{-k-2}, (nt)^{-p}], \end{cases}$$

where A is independent of t and n, and Δ is the usual difference operator. It easily follows from (2.3) that for $u>0$, $\rho>-1$,

$$(2.4)\qquad \sum_{m=1}^{\infty} m^\rho |\Delta^{a+1}\gamma_\alpha(mu)| < Au^{\alpha-\rho}\ (\alpha>\rho-2, a>\rho+1),$$

where A is independent of u.

We require the following three lemmas.

Lemma 1. *The summability* $(C;\alpha,\beta)(R)$ *of* $S[\phi]$ *depends only on the behaviour of* $\phi(u, v)$ *in an arbitrary neighbourhood* $(0, 0; \delta, \delta)$ $(0<\delta<\pi)$ *of the origin.*

This is due to Herriot [5].

Lemma 2. *If* $\alpha>a\geqq 0$, $\beta>b\geqq 0$ *and if* $0<\delta\leqq\pi$ *then for each* $\lambda\geqq 1$,

$$(2.5)\quad \limsup_{\mu\to\infty}\left\{\lambda - \limsup_{(m,n)} \int_0^\delta u^a |K_\alpha^{(a)}(m,u)|\, du \int_0^{u\mu^{-1}} v^b |K_\beta^{(b)}(n,v)|\, dv\right\} = 0.$$

Proof. Denoting the integral in (2.5) by J, and choosing (as we clearly may) $\mu\geqq\lambda$, we have

$$J = \int_0^{m^{-1}} du \int_0^{u\mu^{-1}} dv + \int_{m^{-1}}^{\mu n^{-1}} du \int_0^{u\mu^{-1}} dv + \int_{\mu n^{-1}}^{\delta} du \int_0^{n^{-1}} dv$$
$$+ \int_{\mu n^{-1}}^{\delta} du \int_{n^{-1}}^{u\mu^{-1}} dv = \sum_{r=1}^{4} J_r$$

(say).

By (2.1),

$$J_1 \leqq Am^{a+1}n^{b+1}\int_0^{m^{-1}} u^a du \int_0^{u\mu^{-1}} v^b dv = \frac{Am^{a+1}n^{b+1}}{m^{a+b+2}\mu^{b+1}} \leqq A\left(\frac{\lambda}{\mu}\right)^{b+1},$$

whenever $\lambda^{-1}\leqq mn^{-1}\leqq\lambda$, so that

$$(2.6)\qquad \limsup_{\mu\to\infty}\,(\lambda - \limsup_{(m,n)} J_1) = 0.$$

Next, using (2.1) again,

$$J_2 = An^{b+1}\int_{m^{-1}}^{\mu n^{-1}}(m^{-1}u^{-2} + m^{a-\alpha}u^{a-\alpha-1})du\int_0^{u\mu^{-1}} v^b dv$$
$$= \frac{An^{b+1}}{\mu^{b+1}}\left\{m^{-1}\int_{m^{-1}}^{\mu n^{-1}} u^{b-1}du + m^{a-\alpha}\int_{m^{-1}}^{\mu n^{-1}} u^{a+b-\alpha}du\right\}$$
$$= J_{21} + J_{22}$$

(say), where, if $\lambda^{-1} \leqq mn^{-1} \leqq \lambda$,

$$J_{21} \leqq \begin{cases} A\left(\dfrac{\lambda}{\mu}\right), & b \neq 0, \\ A\dfrac{\lambda}{\mu}\log \lambda\mu, & b = 0, \end{cases}$$

and

$$J_{22} \leqq \begin{cases} A\left(\dfrac{\lambda}{\mu}\right)^{\alpha-a} + A\left(\dfrac{\lambda}{\mu}\right)^{b+1}, & a + b - \alpha \neq -1, \\ A\left(\dfrac{\lambda}{\mu}\right)^{\alpha-a}\log \lambda\mu, & a + b - \alpha = -1, \end{cases}$$

so that in any case,

$$\limsup_{\mu\to\infty}\,(\lambda - \limsup_{(m,n)} J_2) = 0. \tag{2.7}$$

Next, by (2.1),

$$J_3 \leqq An^{b+1}\int_{\mu n^{-1}}^{\delta}(m^{-1}u^{-2} + m^{a-\alpha}u^{a-\alpha-1})du\int_0^{n^{-1}} v^b dv$$
$$\leqq A\frac{\lambda}{\mu} + A\left(\frac{\lambda}{\mu}\right)^{\alpha-a},$$

if $\lambda^{-1} \leqq mn^{-1} \leqq \lambda$ so that

$$\limsup_{\mu\to\infty}\lambda - \limsup_{(m,n)} J_3 = 0. \tag{2.8}$$

Finally by (2.1),

$$J_4 \leqq A\int_{\mu n^{-1}}^{\infty}(m^{-1}u^{-2} + m^{a-\alpha}u^{a-\alpha-1})du\int_{n^{-1}}^{\infty}(n^{-1}v^2 + n^{b-\beta}v^{b-\beta-1})dv$$
$$\leqq A\frac{\lambda}{\mu} + A\left(\frac{\lambda}{\mu}\right)^{\alpha-a},$$

if $\lambda^{-1} \leqq mn^{-1} \leqq \lambda$, so that

$$\limsup_{\mu\to\infty}\,(\lambda - \limsup_{(m,n)} J_4) = 0. \tag{2.9}$$

Combining (2.6)–(2.9) we obtain the required result.

LEMMA 3. *If* $a>\alpha+1\geqq 1$, $b>\beta+1\geqq 1$, *then for every* $\lambda\geqq 1$,

$$\limsup_{\mu\to\infty}\left\{(\lambda - \limsup_{(u,v)} \sum_{m=1}^{\infty} m^{\alpha}\left|\Delta^{\alpha+1}\gamma_a(mu)\right| \sum_{n<m\mu^{-1}} n^{\beta}\left|\Delta^{\beta+1}\gamma_b(nv)\right|\right\} = 0. \tag{2.10}$$

Since the proof of this lemma is quite similar to the proof of Lemma 2 we shall just outline the procedure.

The series in (2.10) is, by (2.3), absolutely convergent, for fixed positive u and v and we denote its sum by $S(u, v)$. We then choose $\lambda(\geqq 1)$ freely and then regard it fixed and take (as we clearly may) $\mu\geqq\lambda$. We then write

$$S(u, v) = \sum_{m=1}^{[u^{-1}]}\sum_{n=1}^{[m\mu^{-1}]} + \sum_{m=[u^{-1}]+1}^{[\mu v^{-1}]}\sum_{n=1}^{[m\mu^{-1}]} + \sum_{m=[\mu v^{-1}]+1}^{\infty}\sum_{n=1}^{[v^{-1}]} + \sum_{m=[\mu v^{-1}]+1}^{\infty}\sum_{n=[v^{-1}]+1}^{[m\mu^{-1}]}$$
$$= \sum_{r=1}^{4} S_r$$

(say). Using (2.3) (as we used (2.1) to estimate the J_r in the proof of Lemma 2) we can show that each of the S_r satisfy a relation

$$S_r \leqq f_r\left(\frac{\lambda}{\mu}\right)$$

for every positive u and v satisfying $\lambda^{-1}\leqq uv^{-1}\leqq\lambda$, and that $f_r(\lambda/\mu)\to 0$ as $\mu\to\infty$. This is sufficient to establish (2.10).

3. **Proof of Theorem** 1. We suppose throughout that the conditions $\alpha\geqq 1$, $\beta\geqq 1$, $\alpha>a\geqq 0$, $\beta>b\geqq 0$, are satisfied, and (without loss of generality) that $s=0$. We also suppose that a and b are integers, the proof may be completed in the general case by standard methods.

It is easily shown that

$$\sigma_{mn}^{\alpha,\beta} = \frac{1}{\pi^2}\int_0^{\pi}\int_0^{\pi}\phi(u, v)K_{\alpha}(m, u)K_{\beta}(n, v)\,du\,dv. \tag{3.1}$$

If we split the range of integration on the right of (3.1) into $(0, 0; \delta, \delta)$, $(0, \delta; \delta, \pi)$, $(\delta, 0; \pi, \delta)$, $(\delta, \delta; \pi, \pi)$; $(0<\delta<\pi)$ then it follows from Lemma 1 that the corresponding integrals over the last three ranges have restricted limit zero as $(m, n)\to(\infty, \infty)$. Hence it is sufficient to show that

$$\left\{\limsup_{\delta\to+0}\lambda - \limsup_{(m,n)}\left|\int_0^{\delta}\int_0^{\delta}\phi(u, v)K_{\alpha}(m, u)K_{\beta}(n, v)\,du\,dv\right|\right\} = 0, \tag{3.2}$$

for each $\lambda\geqq 1$.

Denoting the integral in (3.2) by J we have, on integrating by parts,

$$J = \sum_{r=1}^{a} \sum_{s=1}^{b} (-)^{r+s} \delta^{r+s} (r!s!)^{-1} K_\alpha^{(r-1)}(m, \delta) K_\beta^{(s-1)}(n, \delta) \phi_{r,s}(\delta, \delta)$$

$$+ (-)^{a-1}(a!)^{-1} \sum_{s=1}^{b} (-)^{s-1} \delta^s (s!)^{-1} K_\beta^{(s-1)}(n, \delta) \int_0^\delta \phi_{a,s}(u, \delta) u^a K_\alpha^{(a)}(u) du$$

$$+ (-)^{b-1}(b!)^{-1} \sum_{r=1}^{a} (-)^{r-1} \delta^r (r!)^{-1} K_\alpha^{(r-1)}(m, \delta) \int_0^\delta \phi_{r,b}(\delta, v) v^b K_\beta^{(b)}(v) dv$$

$$+ (-)^{a+b}(a!b!)^{-1} \int_0^\delta \int_0^\delta \phi_{a,b}(u, v) u^a v^b K_\alpha^{(a)}(m, u) K_\beta^{(b)}(n, v) du dv$$

$$= \sum_{r=1}^{4} J_r$$

(say).

It follows immediately from (2.1) that, for any positive δ,

$$J_1 = \sum_{r=1}^{a} \sum_{s=1}^{b} O(m^{-1} + m^{r-1-\alpha}) O(n^{-1} + n^{s-1-\beta}) = o(1),$$

as $(m, n) \to (\infty, \infty)$ in any manner.

Next, using (2.1) again, for any positive δ,

$$J_2 = \sum_{s=1}^{b} O(n^{-1} + n^{s-1-\beta}) O \left\{ m^{a+1} \int_0^{m^{-1}} | \phi_{a,s}(u, \delta) | u^a du \right.$$

$$\left. + \int_{m^{-1}}^{\delta} | \phi_{a,s}(u, \delta) | (m^{-1} u^{-2} + m^{a-\alpha} u^{a-\alpha-1}) du \right\}$$

$$= O(n^{-1} + n^{b-\beta-1}) o(m) = o(1)(R),$$

as $(m, n) \to (\infty, \infty)$.

Similarly $J_3 = o(1)(R)$ as $(m, n) \to (\infty, \infty)$, for any positive δ, and hence to establish (3.2) it is sufficient to show that

$$\limsup_{\delta \to +0} \left(\lambda - \limsup_{(m,n)} | J_4 | \right) = 0, \tag{3.3}$$

for arbitrary $\lambda \geqq 1$.

We now choose λ ($\geqq 1$) freely and then regard it as fixed. Choosing $\mu > \lambda$ we write

$$J_4 = \left\{ \iint_{\substack{0<u<\delta, 0<v<\delta \\ \mu^{-1}<uv^{-1}<\mu}} + \int_0^\delta du \int_0^{u\mu^{-1}} dv + \int_0^\delta dv \int_0^{v\mu^{-1}} du \right\} = \sum_{r=1}^{3} I_r$$

(say).

Since $\phi_{a,b}(u, v)$ is bounded in $(0, 0; \delta, \delta)$ for small δ it follows that there exists K such that

$$\limsup_{\mu\to\infty}\ \limsup_{\delta\to+0}\left(\lambda - \lim_{(m,n)} \left| I_2 + I_3\right|\right) \leqq K \limsup_{\mu\to\infty}\Bigg[\lambda - \limsup_{(m,n)} \cdot\left\{\int_0^{\pi} du \int_0^{u\mu^{-1}} dv + \int_0^{\pi} dv \int_0^{v\mu^{-1}} du\right\} u^a v^b K_\alpha^{(a)}(m, u) K_\beta^{(b)}(n, v)\Bigg] = 0,$$

by Lemma 2.

Finally,

$$\limsup_{\mu\to\infty}\ \limsup_{\delta\to+0}\left(\lambda - \limsup_{(m,n)} \left| I_1\right|\right) \leqq \limsup_{\mu\to\infty}\ \limsup_{\delta\to+0}\Bigg[\lambda - \limsup_{(m,n)} \cdot\left\{\mu - \sup_{0<u\leqq\delta;\,0<v\leqq\delta} \left|\phi_{a,b}(u, v)\right| \int_0^{\pi} u^a \left|K_\alpha^{(a)}(m, u)\right| du \int_0^{\pi} v^b \left|K_\beta^{(b)}(n, v)\right| dv\right\}\Bigg] = 0,$$

by (2.2) and the fact that $\mu - \lim_{(u,v)} \left|\phi_{a,b}(u, v)\right| = 0$.

Thus

$$\lim_{\mu\to\infty}\ \limsup_{\delta\to+0}\left(\lambda - \limsup_{(m,n)} \left| J_4\right|\right) \leqq \limsup_{\mu\to\infty}\ \limsup_{\delta\to+0}\left[\lambda - \limsup_{(m,n)}\left\{\left|I_1\right| + \left|I_2 + I_3\right|\right\}\right] = 0,$$

and this establishes (3.3).

4. **Proof of Theorem** 2. We suppose throughout that $a-2>\alpha\geqq 0$, $b-2>\beta\geqq 0$ and (without loss of generality) that $s=0$. We also suppose that α and β are integers, the proof may be completed in the general case by standard methods.

Since $a\geqq 1$, $b\geqq 1$ it follows that $(u-x)^{a-1}$, $(v-y)^{b-1}$ are of bounded variation in $0\leqq x\leqq u$ and $0\leqq y\leqq v$ respectively and hence from (1.1), by a straightforward extension of a well-known result [6, p. 583], that for $u>0$, $v>0$

$$\begin{aligned}\phi_{a,b}(u, v) &= abu^{-a}v^{-b}\int_0^u (u - x)^{a-1}dx \int_0^v (v - y)^{b-1}\phi(x, y)dy \\ &= \sum_{m,n=0}^{\infty} a_{mn} a u^{-a}\int_0^u (u - x)^{a-1}\cos mn\, dx\, b v^{-b}\int_0^v (v - y)^{b-1}\cos xy\, dy \\ &= \sum_{m,n=0}^{\infty} a_{mn}\gamma_a(mu)\gamma_b(nv).\end{aligned}$$

By repeated application of Abel's lemma we obtain

$$\phi_{a,b}(u,v) = \lim_{N\to\infty}\left\{\sum_{m,n=0}^{N} S_{mn}^{\alpha,\beta}\Delta^{\alpha+1}\gamma_a(mu)\Delta^{\beta+1}\gamma_b(nv)\right.$$

$$+\sum_{m=0}^{N}\sum_{s=0}^{\beta} S_{mN-1}^{\alpha,s}\Delta^{\alpha+1}\gamma_a(mu)\Delta^{s}\gamma_b(Nv)$$

$$+\sum_{n=0}^{N}\sum_{r=0}^{\alpha} S_{N-1n}^{r,\beta}\Delta^{r}\gamma_a(Nu)\Delta^{\beta+1}\gamma_b(nv)$$

$$\left.+\sum_{r=0}^{\alpha}\sum_{s=0}^{\beta} S_{N-1N-1}^{r,s}\Delta^{r}\gamma_a(Nu)\Delta^{s}\gamma_b(Nv)\right\}$$

$$= \lim_{N\to\infty}\sum_{i=1}^{4} T_i$$

(say).

Since $a_{mn}=o(1)$ as $(m, n)\to(\infty, \infty)$ it follows that $S_{mn}^{r,s}=o(m^{r+1}n^{s+1})$ as $(m, n)\to(\infty, \infty)$ for $r=0, 1, \cdots, s=0, 1, \cdots$ and hence, from (2.3), that for fixed positive u and v

$$T_4 = \sum_{r=0}^{\alpha}\sum_{s=0}^{\beta} O(N^{r+s+2})O(N^{-r-2}+N^{-a})O(N^{-s-2}+N^{-b})$$

$$= o(1),$$

as $N\to\infty$.

Next, since $S_{mN-1}^{\alpha,s}=o(m^{\alpha+1}N^{s+1})$ it follows, using (2.3) and (2.4), that, for fixed positive u and v,

$$T_2 = O\left\{(N^{-1}+N^{\beta-b+1})\sum_{m=1}^{N} m^{\alpha}\left|\Delta^{\alpha+1}\gamma_a(mu)\right|\right\}$$

$$= O(N^{-1})O(1) = o(1)$$

as $N\to\infty$.

Similarly $T_3=o(1)$ as $N\to\infty$ for fixed positive u and v so that, for $u>0$, $v>0$,

$$\phi_{a,b}(u,v) = \sum_{m,n=0}^{\infty} S_{mn}^{\alpha,\beta}\Delta^{\alpha+1}\gamma_a(mu)\Delta^{\beta+1}\gamma_b(nv)$$

$$= \sum_{m=0}^{N}\sum_{n=0}^{N} + \sum_{m=N+1}^{\infty}\sum_{n=0}^{N} + \sum_{m=0}^{N}\sum_{n=N+1}^{\infty} + \sum_{m=N+1}^{\infty}\sum_{n=N+1}^{\infty}$$

$$= \sum_{r=1}^{4} U_r$$

(say).

It follows immediately from (2.3) that for any positive N,

$$\begin{aligned} |U_1| &= O\left(u^{\alpha+1} v^{\beta+1} \sum_{m=0}^{N} \sum_{n=0}^{N} |S_{mn}^{\alpha,\beta}|\right) \\ &= o(1), \end{aligned}$$

as $(u, v) \to (+0, +0)$ in any manner.

Next since $a_{mn} \to 0$ as $m \to \infty$ for each $n \geqq 0$ it follows that $S_{mn}^{\alpha,\beta} = o(m^{\alpha+1})$ for $n = 0, 1, \cdots$ and hence, using (2.3), that for each positive N,

$$\begin{aligned} |U_2| &= O(v^{\beta+1}) \sum_{m=1}^{\infty} o(m^{\alpha+1}) \,|\Delta^{\alpha+1}\gamma_a(mu)| \\ &= O(v^{\beta+1}) o(u^{-1}) \qquad \text{(by 2.4)} \\ &= o(1)(R) \end{aligned}$$

as $(u, v) \to (+0, +0)$.

In a similar way we can show that $U_3 = o(1)(R)$ as $(u, v) \to (+0, +0)$. Hence in order to establish the theorem it is sufficient to show that for each $\lambda \geqq 1$

$$\limsup_{N \to \infty} (\lambda - \limsup_{(u,v)} |U_4|) = 0. \tag{4.1}$$

To this end we first choose λ freely and then regard it as fixed. We then choose $\mu > \lambda$ and write

$$\begin{aligned} U_4 &= \left\{ \sum_{\substack{m>N, n>N \\ \mu^{-1} \leqq mn^{-1} \leqq \mu}} + \sum_{\substack{m>N, n>N \\ mn^{-1} > \mu}} + \sum_{\substack{m>N, n>N \\ mn^{-1} < \mu^{-1}}} \right\} S_{mn}^{\alpha,\beta} \Delta^{\alpha+1} \gamma_a(mu) \Delta^{\beta+1} \gamma_b(nv) \\ &= \sum_{r=1}^{3} V_r \end{aligned}$$

(say).

Since $\sigma_{mn}^{\alpha,\beta}$ is bounded for large m and n it follows that there exists K such that

$$\begin{aligned} &\limsup_{\mu \to \infty} \limsup_{N \to \infty} \left(\lambda - \limsup_{(u,v)} |V_2 + V_3|\right) \leqq K \limsup_{\mu \to \infty} \Bigg[\lambda - \limsup_{(u,v)} \\ &\qquad \cdot \left\{ \sum_{m=1}^{\infty} \sum_{n < m\mu^{-1}} + \sum_{n=1}^{\infty} \sum_{m < n\mu^{-1}} \right\} m^\alpha n^\beta \,|\Delta^{\alpha+1}\gamma_a(mu)| \,|\Delta^{\beta+1}\gamma_b(nv)| \Bigg] \\ &= 0, \end{aligned}$$

by Lemma 3.

Finally,

$$\limsup_{\mu\to\infty}\limsup_{N\to\infty}\left(\lambda-\lim_{(u,v)}|V_1|\right)\leqq\limsup_{N\to\infty}\Big[\lambda-\limsup_{(u,v)}$$

$$\cdot\left\{\lambda-\sup_{m>N,n>N}|\sigma_{mn}^{\alpha,\beta}|\sum_{m=1}^{\infty}m^{\alpha}|\Delta^{\alpha+1}\gamma_a(mu)|\sum_{n=1}^{\infty}n^{\beta}|\Delta^{\beta+1}\gamma_b(nv)|\right\}\Big]$$

$$=0,$$

by (2.4) and the fact that $\lambda-\lim_{(m,n)}|\sigma_{mn}^{\alpha,\beta}|=0$.

Consequently

$$\limsup_{\mu\to\infty}\limsup_{N\to\infty}(\lambda-\limsup_{(u,v)}|U_4|)$$

$$\leqq\limsup_{\mu\to\infty}\limsup_{N\to\infty}\left[\lambda-\limsup_{(u,v)}\{|V_1|+|V_2+V_3|\}\right]$$

$$=0,$$

which establishes (4.1).

5. **An example.** In view of the single Fourier series theorem on which Theorem 1 is based (cf. [1, Theorem 1]) and in view of the facts (i) that the series in (1.1) is summable almost everywhere in $(0, 0; \pi, \pi)$ to $\phi(u, v)$ and (ii)

$$abs^{-a}t^{-b}\int_0^s\int_0^t(s-x)^{a-1}(t-y)^{b-1}\phi(u+x, v+y)dxdy\to\phi(u, v)(R)$$

as $(s, t)\to(+0, +0)$ for almost every $(u, v)\in(0, 0; \pi, \pi)$; it is attractive to conjecture that $\phi(u, v)\to s(C; a, b)(R)$ is alone sufficient for the conclusion of Theorem 1. That this conjecture is false, i.e., that the boundedness condition on $\phi_{a,b}(u, v)$ in Theorem 1 cannot be entirely removed, is shown by the following example.

Let $\phi(u, v)$ be even and periodic with period 2π and such that

$$\phi(r\cos\theta, r\sin\theta)=\begin{cases}r^{-2}e^{-\theta r^{-1}} & 0<r<\pi, 0\leqq\theta\leqq\pi/2,\\ 0 & \text{elsewhere in } (0, 0; \pi, \pi).\end{cases}$$

It is easily verified that $\phi(u, v)\in L(0, 0; \pi, \pi)$ and that $\phi(u, v)\to 0(R)$ as $(u, v)\to(+0, +0)$.

Since $\phi(u, v)$ is positive and since $\sin^2 mu/2\geqq(4m^2/\pi^2)\sin^2 u/2$ for $0<u\leqq 1/2m$ it follows that

$$\sigma_{mm}^{1,1}>\frac{1}{\pi^2}\int_0^{1/2m}\int_0^{u}\phi(u, v)\frac{\sin^2 mu/2}{m\sin^2 u/2}\frac{\sin^2 mv/2}{m\sin^2 v/2}dudv$$

$$\geqq\frac{16m^2}{\pi^6}\int_0^{1/2m}r^{-1}dr\int_0^{\pi/4}e^{-\theta r^{-1}}d\theta$$

$$=\frac{16m^2}{\pi^6}\int_0^{1/2m}(1-e^{-\pi/4r})dr>\frac{8m^2}{\pi^6}\int_0^{1/2m}dr\to\infty$$

as $m \to \infty$. Consequently $\sigma_{mm}^{1,1} \to \infty$ as $m \to \infty$ and a fortiori $S[\phi]$ is not summable $(C; 1, 1)$ (R) even though $R - \lim_{(u,v) \to (+0,+0)} \phi(u, v)$ exists.

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